On Optimal Timer-Based Distributed Selection For Rate-Adaptive Multi-user Diversity Systems

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***Abstract*—We develop an optimal, distributed, and low feedback timer-based selection scheme to enable next generation rate-adaptive wireless systems to exploit multi-user diversity. In our scheme, each user sets a timer depending on its signal to noise ratio (SNR) and transmits a small packet to identify itself when its timer expires. When the SNR-to-timer mapping is monotone non-decreasing, timers of users with better SNRs expire earlier. Thus, the base station (BS) simply selects the first user whose timer expiry it can detect, and transmits data to it at as high a rate as reliably possible. However, timers that expire too close to one another cannot be detected by the BS due to collisions. We characterize in detail the structure of the SNR-to-timer mapping that optimally handles these collisions to maximize the average data rate. We prove that the optimal timer values take only a discrete set of values, and that the rate adaptation policy strongly influences the optimal scheme's structure. The optimal average rate is very close to that of ideal selection in which the BS always selects highest rate user, and is much higher than that of the popular, but ad hoc, timer schemes considered in the literature.**

I. Introduction

Multi-user diversity and rate adaptation [1] are two techniques that have led to significant improvements in the rates delivered by third generation cellular systems [2], wireless local area networks [3] and ad hoc networks [4]. Multi-user diversity exploits the fact that among several users, the signal to noise ratio (SNR) of at least one user is high with a high probability. Rate adaptation then enables the BS to transmit data at as high a rate as reliably possible.

The first step in exploiting multi-user diversity is the selection of the user with the highest SNR. This is typically achieved by a polling scheme that makes each user periodically transmit its channel gains to the BS, which then selects the best one among them [2]. However, the time and bandwidth resources consumed by polling increase as the number of users increases. An alternate option is the use of distributed mechanisms to select a user [5]–[7]. These mechanisms are attractive because they are considerably faster than the polling scheme and also scale remarkably well with the number of users. They can also be adapted to accommodate fairness and quality of service considerations.

A popular distributed selection mechanism is the timer scheme [6], [8], [9]. In it, each user sets its timer as a function of its SNR, and transmits a small packet containing its identity when its timer expires. When the SNR-to-timer mapping is a monotone non-increasing (MNI) function, the timer of the best user always expires first. No feedback from the BS is required during the selection process. All the BS does is select the user responsible for the first timer expiry it observes, and transmit to it with an appropriate rate. This process of selection followed by data transmission to the selected user is repeated.

In practice, for the selection to be successful, it is necessary that no other timer expires within a vulnerability time interval Δ after the expiry of the best user's timer [6], [9], [10]. Otherwise, a collision occurs, and the system fails to select the best user. Δ typically includes the maximum propagation and detection delays between all nodes. Even though Δ is typically one to two orders of magnitude smaller than a packet transmission duration, it is the collisions that limit the ability of the system to exploit multi-user diversity. Increasing the timer values to reduce the probability of a collision is not desirable since it also increases the total time required to select. Therefore, the SNR-to-timer mapping needs to be judiciously chosen so as to maximize overall rate. In general, finding the optimal mapping is a difficult problem because of the infinitely many possible MNI functions. For example, only ad hoc mappings were proposed in [6], [8].

Another important factor that determines the overall rate is the selection policy. For example, in the event of a collision that involves the best user, the BS can declare an outage, as was assumed in [6], [9]. However, this is pessimistic since even the rate to the kth best user may be significant depending on its instantaneous channel state. *Thus, a better pragmatic policy is to make the BS wait for the first timer that it can reliably detect – even if collisions precede it – and select the user responsible for it. Even when all the timers collide, all is not lost; the BS just randomly selects one user.*

In this paper, we consider a rate-adaptive multi-user system in which the BS selects a user using the distributed timer mechanism. We derive the optimal SNR-to-timer mapping that maximizes the overall system rate. We first show that the system rate is maximized when the SNR-to-timer mapping is discrete, *i.e.*, the timer set by any user expires only at times $0, \Delta, \ldots$, or $N\Delta$, where $N = \left\lfloor \frac{T_{\max}}{\Delta} \right\rfloor$, $T_{\max}$ is the maximum selection duration, and $\lfloor . \rfloor$ is the floor operation. We then show that all the optimal intervals can, in fact, be written in terms of at most $M-1$ integers and one real number in $[0, 1]$. This enables the optimal policy to be easily computed. We also show that the structure of the optimal timer scheme is closely inter-linked with the rate adaptation rule used by the BS.

The paper is organized as follows. Section II describes the system model. The optimal timer scheme and its performance

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are analyzed in Sec. III. Simulation results in Sec. IV are followed by our conclusions in Sec. V.

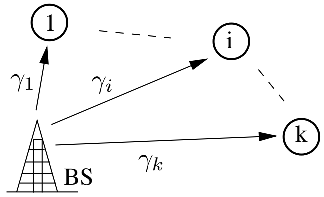

Fig. 1. A system consisting of a BS and k mobile users. Each user i sets its timer depending on its SNR γ_i.

II. System Model

We consider a system with k users and a BS. The channels from the users to the BS are assumed to be independent and identically distributed (i.i.d.) block fading channels. Our analysis applies to channels with arbitrary fading distributions. Using standard order statistics notation, we order the power gains as $\gamma_{(1)} \geq \gamma_{(2)} \geq \cdots \geq \gamma_{(k)}$, where (i) denotes the index of the ith best user. The BS first selects the user using the timer scheme, and then transmits data to it. We first model below the data rate adaptation policy, and then the timer scheme.

A. Rate Adaptation

Rate adaptation is done based on the SNR of the selected user such that the error probability of data transmission does not exceed a pre-specified value. It is specified by a set of rates $R_M < R_{M-1} < \cdots < R_1$ and rate adaptation thresholds $0 = \Gamma_M < \Gamma_{M-1} < \cdots < \Gamma_0 = \infty$ such that the rate of a user with channel gain γ_i is R_i when $\Gamma_i \leq \gamma_i < \Gamma_{i-1}$. This includes the oft-encountered special case of $R_M = 0$.

This is best understood by an example that adapts the size of the MPSK constellation. A tight bound for the bit error probability, BER_Q, of MPSK of constellation size Q is [1]

$$\mathrm{BER}_Q = 0.2\exp\left(-1.5\gamma/(Q-1)\right), \ Q \geq 4. \qquad (1)$$

Inverting this equation, we find that a data rate of R_i bits/symbol can be reliably supported so long as $\gamma \geq \frac{(2^{R_i}-1)}{1.5}\log_e\left(\frac{1}{5P_b}\right)$, where P_b is the target bit error rate. Our framework applies equally well to adaptation based on MQAM or coded modulation schemes, whose thresholds can be similarly tabulated using analysis or simulations [1], [11].

B. Timer-Based Selection

In order to handle all SNR probability distributions, we first normalize the SNRs, without loss of generality. Let $\mu_i = F(\gamma_i)$, where $F(.)$ is the cumulative distribution function (CDF) of the SNR. *Now, regardless of the distribution of the SNR, μ_i is a random variable (RV) that is uniformly distributed in the interval $[0,1]$ [12].* We shall refer to μ_i as the *metric* of User i. The metric and SNR are equivalent for the purpose of selection because the CDF is a monotonically non-decreasing function. The corresponding CDF-normalized values of the adaptation thresholds are

$$\Gamma_i^{\mathrm{eq}} = F(\Gamma_i), \quad \text{for } 0 \leq i \leq M. \qquad (2)$$

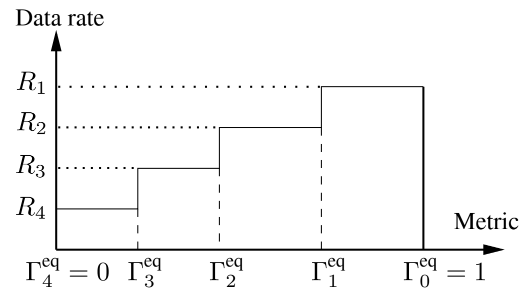

Fig. 2. Rate adaptation as a function of the metric when $M = 4$.

For example, for Rayleigh fading, $\Gamma_i^{\mathrm{eq}} = 1 - e^{-\Gamma_i/\overline{\gamma}}$, where $\overline{\gamma}$ is the average SNR. These are illustrated in Fig. 2.

The BS allocates a time of $T_{\max}$ for selecting a user. User i sets its timer as a function of its metric, μ_i, as $T_i = \xi(\mu_i)$. Here, $\xi(.)$ is the metric-to-timer mapping MNI function, which we shall optimize, with a domain $[0,1]$ and a range $[0,\infty)$. User i transmits a small 'timer packet' to the BS when its timer expires if $T_i < T_{\max}$. If two timers expire within a duration less than Δ of each other, their timer packets collide and cannot be decoded by the BS. Let $N = \left\lfloor \frac{T_{\max}}{\Delta} \right\rfloor$.

As mentioned, the BS selects the user from which it first receives a timer packet successfully. For example, the BS will select User (1) if the packet of User (2) does not collide with it. Else, in the event of a collision between users (1) and (2), it selects User (3) in case the timer of User (4) does not collide with it, and so on. If no user has got selected by time $T_{\max}$, then the BS just selects a user randomly. The BS then communicates with the user that it has selected and transmits data to it – with a rate as determined by Sec. II-A.

III. Optimal Timer Scheme Characterization

Let $\overline{R}_N$ denote the average rate achieved by the BS when it transmits to the selected user and $\overline{R}_{\text{rnd}}$ be the average rate obtained just by random selection.[1] We now derive the timer scheme that maximizes $\overline{R}_N$ given the system parameters N amd M. The Lemma below shows the intuitive result that since a BS can choose a user randomly at time $T_{\max}$, only users whose rates strictly exceed $\overline{R}_{\text{rnd}}$ should participate in the selection process. The proof is omitted to conserve space.

Lemma *1:* Let P be the largest integer such that $R_P > \overline{R}_{\text{rnd}}$. In an optimal timer scheme, only timers of users whose rate is greater than or equal to R_P expire before $T_{\max}$. ■

Since $\overline{R}_{\text{rnd}} \geq R_M$, strictly fewer than M rates need to be considered in designing the selection scheme. We now prove that an optimal metric-to-timer mapping $f(.)$ is discrete.

[1] The role of the subscript $N = \left\lfloor \frac{T_{\max}}{\Delta} \right\rfloor$ in $\overline{R}_N$ will become clear after Theorem 1. In our average rate calculations, we do not account for the time overhead ($T_{\max}$) of selection because our goal is to find the optimal timer scheme given any $T_{\max}$. An overall system optimization that also accounts for and optimizes $T_{\max}$ is beyond the scope of this paper. It has been considered, for example, in [13].

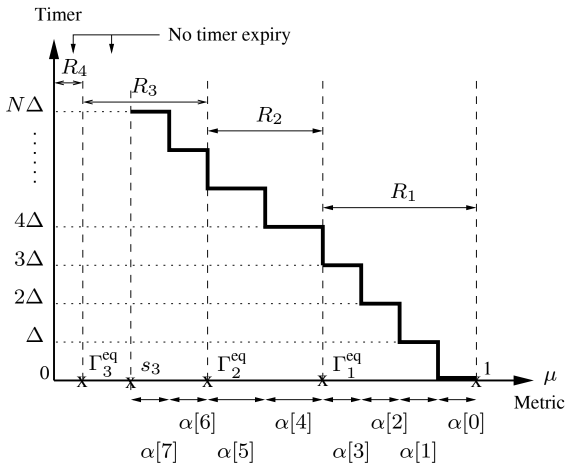

Fig. 3. Illustration of the optimal timer mapping for $N = 8$ slots and $M = 4$ rates. It shows that $r_1 = 4$ equal length intervals occupy the entire interval $[\Gamma_1^{\rm eq}, 1)$ for which the rate is R_1. Similarly, $r_2 = 2$ and $r_3 = 2$. Furthermore, $\Gamma_3^{\rm eq} \leq s_3 < \Gamma_2^{\rm eq}$.

Theorem *1:* An optimal mapping that maximizes the average rate maps μ into $N+1$ discrete timer values $\{0, \Delta, \ldots, N\Delta\}$, where $N = \left\lfloor \frac{T_{\max}}{\Delta} \right\rfloor$.

Proof: The proof is given in Appendix A. ∎

Thus, the longer the selection duration, the more the number of intervals $(N+1)$ available for timer expiry. Since the optimal mapping, $f^*(\mu)$, is discrete, let $\alpha^*[j]$ denote the length of the jth interval in which the metric is mapped into the timer $j\Delta$ $(0 \leq j \leq N)$. Therefore,

$$f^*(\mu) = \begin{cases} 0, \ 1 \leq \mu < 1 - \alpha^*[0] \\ i\Delta, \ 1 - \sum_{j=0}^{i} \alpha^*[j] \leq \mu < 1 - \sum_{j=0}^{i-1} \alpha^*[j], \\ \qquad\qquad\qquad\qquad\qquad \text{for } 1 \leq i \leq N \\ T_{\max} + \epsilon, \ \text{otherwise } (\textit{i.e.}, \text{no timer expiry}) \end{cases}, \quad (3)$$

where $\epsilon > 0$ is an arbitrary constant. Thus, for a user whose metric lies in the 0th interval $[1-\alpha^*[0], 1)$, the timer expires at 0; for a user whose metric lies in the 1st interval $[1-\alpha^*[1]-\alpha^*[2], 1-\alpha^*[1])$, the timer expires at Δ, and so on. Timers of users with metrics below $1 - \sum_{i=0}^{N} \alpha^*[i]$ do not expire.

While a similar discreteness result occurs in [9], there are several differences between the model and results in this paper and those in [9]. The model in [9] is much simpler because the BS just declares an outage once the timer of the best user is involved in a collision. Furthermore, no application of selection such as rate adaptation is considered. As we shall see, this strongly influences the structure of the optimal mapping.

We have reduced the problem of finding $f(\mu)$ over the enormous space of all positive-valued MNI functions to a problem of finding just the lengths of $(N+1)$ intervals, which involves a search over $[0,1]^{N+1}$. Even this search problem is non-trivial since Δ is often small compared to $T_{\max}$, which makes N often exceed 10. The following observation simplifies the problem further.

Observation: The number of rates in a system that need to be considered, P, is typically much less than the number of timer intervals N. Since $P \ll N$, the adaptation thresholds, $\Gamma_P^{\rm eq}, \ldots, \Gamma_1^{\rm eq}$, lie within a small minority of the $(N+1)$ intervals. Consequently, for many intervals, the rate is often the same for all metrics that belong to an interval. We shall, therefore, assume that this is true for *all* the intervals. In order to guarantee a given error rate, this assumption implies that the rate of a user whose metric lies in the jth interval (of length $\alpha[j]$) is determined by its lower metric limit $1 - \sum_{i=0}^{j} \alpha[i]$. *Since all metrics that belong to an interval now yield the same rate, we can now associate each interval with a single rate.*

Notation: Let $I[j]$ denote the rate when a user from the jth interval is selected; $I[j] = R_i$ if the interval $\alpha[j]$ is a subset of $[\Gamma_i^{\rm eq}, \Gamma_{i-1}^{\rm eq})$. Let x_i denote the number of users whose metrics lie in the ith interval of length $\alpha[i]$. Let $\mathbf{x} = (x_0, \ldots, x_N)$. Define the *selection function* $\mathcal{S}(\mathbf{x})$ as $\mathcal{S}(\mathbf{x}) = i$ if $x_i = 1$ and $x_j \neq 1$ for all $j < i$. Thus, $\mathcal{S}(\mathbf{x}) = i \in \{0, N\}$ if a user from the ith interval gets selected. Therefore, the rate achieved by the timer scheme equals $I[\mathcal{S}(\mathbf{x})]$. When all timers collide, the BS ends up selecting a user randomly; we indicate this by defining for this case: $\mathcal{S}(\mathbf{x}) = N+1$ and $I[N+1] = \overline{R}_{\rm rnd}$.

The probability of occurrence of $\mathbf{x}$ is $\binom{k}{x_0, \ldots, x_N} \left[\prod_{i=0}^{N} (\alpha[i])^{x_i}\right] \left(1 - \sum_{i=0}^{N} \alpha[i]\right)^{k - \sum_{i=0}^{N} x_i}$, where $\binom{k}{x_0, \ldots, x_N}$ is the multinomial function. Thus, the average rate, $\overline{R}_N$, when the BS transmits to the selected user is

$$\overline{R}_N = \sum_{\mathbf{x}} \binom{k}{x_0, \ldots, x_N} \left[\prod_{i=0}^{N} (\alpha[i])^{x_i}\right] \times \left(1 - \sum_{i=0}^{N} \alpha[i]\right)^{k - \sum_{i=0}^{N} x_i} I[\mathcal{S}(\mathbf{x})]. \quad (4)$$

The following result characterizes the optimal lengths of all the $N+1$ intervals.

Theorem *2:* In the optimal timer scheme, there exist $m \leq P$ positive integers $r_1, \ldots, r_m$ such that $\sum_{i=1}^{m} r_i = N+1$ and

$$\begin{aligned} \alpha^*[0] = \ldots = \alpha^*[r_1 - 1] &= \frac{1 - s_1}{r_1}, \\ \alpha^*[r_1] = \ldots = \alpha^*[r_1 + r_2 - 1] &= \frac{s_1 - s_2}{r_2}, \\ \vdots \qquad\qquad & \qquad \vdots \\ \alpha^*\left[\sum_{i=1}^{m-2} r_i\right] = \ldots = \alpha^*\left[\sum_{i=1}^{m-1} r_i - 1\right] &= \frac{s_{m-2} - s_{m-1}}{r_{m-1}}, \\ \alpha^*\left[\sum_{i=1}^{m-1} r_i\right] = \ldots = \alpha^*[N] &= \frac{s_{m-1} - s_m}{r_m}, \end{aligned} \quad (5)$$

where $s_1, \ldots, s_m$ are m real numbers such that $1 > s_1 \geq \Gamma_1^{\rm eq} > s_2 \geq \Gamma_2^{\rm eq} > \cdots \geq \Gamma_{m-1}^{\rm eq} > s_m \geq \Gamma_m^{\rm eq}$.

Proof: The proof is given in Appendix B. ∎

In case $m < P$, even the timers of users whose rates exceed $\overline{R}_{\rm rnd}$ but are less than or equal to R_{m+1} do not expire.

The problem has now been reduced to finding $m \leq P$ positive integers whose sum is N and m real numbers $s_1, \ldots, s_m$ that maximize $\overline{R}_N$. These can be found numerically.

Approximation: To simplify the numerical computations even further, we use the following approximation:

$$s_1 = \Gamma_1^{\text{eq}}, \ldots, s_{m-1} = \Gamma_{m-1}^{\text{eq}}. \tag{6}$$

Note that s_m, which from Theorem 2 equals $1 - \sum_{i=0}^{N} \alpha^* [i]$, is left untouched. As we shall see, the average rate obtained using this approximation is indistinguishable from a brute force search that finds the optimal s_i. It works because, for typical N, the interval lengths are small enough that slightly shifting s_i has a negligible effect on $\overline{R}_N$. The structure of the optimal scheme after using this approximation is shown in Fig. 3. The average rate of the optimal scheme then is

$$\overline{R}_N = \sum_{\mathbf{x}} I\left[\mathcal{S}\left(\mathbf{x}\right)\right] \binom{k}{x_0, \ldots, x_N} \left(\frac{1-\Gamma_1^{\text{eq}}}{r_1}\right)^{\sum_{i=0}^{r_1-1} x_i} \times \left(\frac{\Gamma_1^{\text{eq}} - \Gamma_2^{\text{eq}}}{r_2}\right)^{\sum_{i=r_1}^{r_1+r_2-1} x_i} \cdots \left(\frac{\Gamma_{m-1}^{\text{eq}} - s_m}{r_m}\right)^{\sum_{i=r_1+\cdots+r_{m-1}}^{N} x_i}.$$

The problem has now been reduced to finding one real number and $m \leq P$ nonnegative integers that maximize $\overline{R}_N$. These can be found numerically.

IV. Results

We now study the rate achieved by the optimal timer scheme. For the purpose of illustration, the rate adaptation thresholds are generated using (1) for $R_1 = 4$ bits/symbol (16-PSK), $R_2 = 3$ bits/symbol (8-PSK), $R_3 = 2$ bits/symbol (QPSK), and $R_4 = 0$. The corresponding thresholds in linear scale are: $\Gamma_0 = \infty$, $\Gamma_1 = 17.24$, $\Gamma_2 = 13.93$, $\Gamma_3 = 10.25$, and $\Gamma_4 = 0$. Rayleigh fading is assumed for all channels, which have an average SNR of $\overline{\gamma} = 10$ dB. In this case, the equivalent CDF-normalized thresholds become $\Gamma_0^{\text{eq}} = 1$, $\Gamma_1^{\text{eq}} = 0.99$, $\Gamma_2^{\text{eq}} = 0.92$, $\Gamma_3^{\text{eq}} = 0.65$, and $\Gamma_4^{\text{eq}} = 0$. The average rate with random selection is $\overline{R}_{\text{rnd}} = 0.79$ and $P = 3$ in this case.[2]

Figure 4 plots the rate of the optimal timer scheme as a function of the total number of users, k, when $N = \left\lfloor \frac{T_{\max}}{\Delta} \right\rfloor = 10$. Such a small N is chosen deliberately in order to stress the approximations made in Sec. III. Also plotted for comparison are the rates of: (i) the optimal scheme obtained whose intervals are determined using a brute-force $N+1$ dimensional numerical search, (ii) the hypothetical perfect case in which the best user is always selected, and (iii) the popular inverse metric timer scheme of [6], in which the metric-to-timer mapping is $\xi(\mu) = c/\mu$. In order to provide as fair a comparison as possible, the parameter c is optimized numerically for each set of k and N.

The rate of the optimal scheme obtained from a brute force search is very close to to the one obtained using the approximation of (6). We, therefore, no longer distinguish between the two. Furthermore, its rate is quite close to that of the perfect selection even for N as small as 10, and increases with k. It also substantially outperforms the optimized inverse metric scheme, whose rate does not increase with k. The figure also highlights the role played by the selection policy. For this, it plots the rate of the optimal best user selection scheme of [9], which minimizes the probability that the best user's timer is involved in a collision, but declares an outage if the best user's timer is involved in a collision. We see that our optimal scheme, which is tailored to a more pragmatic selection policy, is different and substantially better.

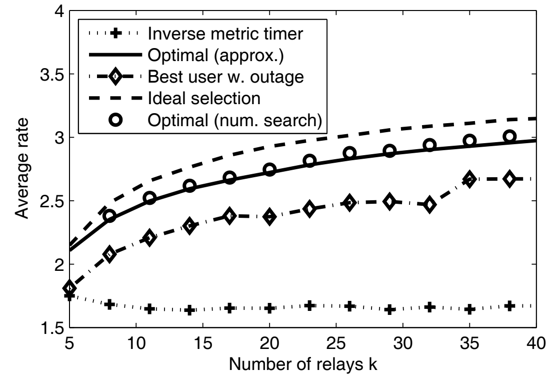

Fig. 4. Average rate (in bits/symbol) as a function of k when $N = \left\lfloor \frac{T_{\max}}{\Delta} \right\rfloor = 10$ ($R_1 = 4$, $R_2 = 3$, $R_2 = 2$, and $R_1 = 0$)

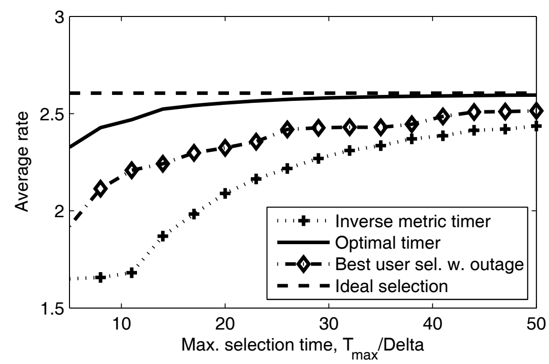

Fig. 5. Average rate (in bits/symbol) as a function of $T_{\max}/\Delta$ when $k = 10$ ($R_1 = 4$, $R_2 = 3$, $R_2 = 2$, and $R_1 = 0$)

Figure 5 plots the rate of the optimal scheme as a function of $N = \left\lfloor \frac{T_{\max}}{\Delta} \right\rfloor$ when there are $k = 10$ users in the system. As expected, as N increases, the rate increases. As in Fig. 4, also plotted are the rates of the optimized inverse metric scheme, the optimal best node selection scheme with outage, and the ideal case where the best node always gets selected. Clearly, the rate of the ideal case does not depend on N. The optimal scheme is as good as perfect selection when $N \geq 25$, and significantly better than the inverse metric and best node selection schemes.

V. Conclusions

We considered a rate-adaptive multi-user diversity system that uses a distributed timer scheme instead of the slow polling

[2]The average SNR, $\overline{\gamma}$, needs to be specified since the values of P, $\overline{R}_{\text{rnd}}$, and Γ_i^{eq} depend on it.

scheme to determine which user the BS transmits to. We used a pragmatic selection policy in which the BS selects the user whose timer it detects first, even if it is not the best user. We showed that, for a given maximum selection duration, the optimal metric-to-timer mapping that maximizes the rate is discrete in nature, and that all the metric intervals can be written in terms of the rate adaptation thresholds. The rate of the optimal timer scheme is quite close to that of the ideal case in which the BS always perfectly selects the best user. The optimal scheme significantly outperforms the popular inverse metric timer scheme – even after the latter is optimized – and the best user selection policy, in which the BS declares an outage in case it the best user's timer is involved in a collision.

APPENDIX

A. Proof of Theorem 1

Let $f(\mu)$ be an optimal MNI metric-to-timer mapping. Consider the new mapping $g(\mu)$:

$$g(\mu) = \begin{cases} \left\lfloor \frac{f(\mu)}{\Delta} \right\rfloor \Delta, & \text{if } f(\mu) \leq T_{\max} \\ f(\mu), & \text{if } f(\mu) > T_{\max} \end{cases}. \tag{7}$$

Therefore, $g(\mu) = 0$, if $0 \leq f(\mu) < \Delta$, $g(\mu) = \Delta$, if $\Delta \leq f(\mu) < 2\Delta$, and so on. Since $f(\mu)$ is MNI, $g(\mu)$ is well-defined and is also MNI. We prove below that the rate obtained using $g(\mu)$ is greater than or equal to that from $f(\mu)$.

For $f(\mu)$, let the k timers take values $T_1, \ldots, T_k$, and let User (i) get selected. Let the corresponding timer values for $g(\mu)$ be $T'_1 = \left\lfloor \frac{T_1}{\Delta} \right\rfloor \Delta, \ldots, T'_k = \left\lfloor \frac{T_k}{\Delta} \right\rfloor \Delta$.

Case 1: $T_{(i+1)}$ also expired before $T_{\max}$ and $i \geq 2$: Since User (i) got selected, timers $T_{(1)}, \ldots, T_{(i-1)}$ collided with one another and $T_{(i-1)}$ and $T_{(i+1)}$ did not collide with $T_{(i)}$. Therefore, $T_{(i+1)} - T_{(i)} \geq \Delta$ and $T_{(i)} - T_{(i-1)} \geq \Delta$. Hence,

$$\left\lfloor \frac{T_{(i+1)}}{\Delta} \right\rfloor \geq \left\lfloor \frac{T_{(i)}}{\Delta} \right\rfloor + 1 \quad \text{and} \quad \left\lfloor \frac{T_{(i-1)}}{\Delta} \right\rfloor \leq \left\lfloor \frac{T_{(i)}}{\Delta} \right\rfloor - 1. \tag{8}$$

Since $T'_{(i-1)} = \left\lfloor \frac{T_{(i-1)}}{\Delta} \right\rfloor \Delta$, $T'_{(i)} = \left\lfloor \frac{T_{(i)}}{\Delta} \right\rfloor \Delta$, and $T'_{(i+1)} = \left\lfloor \frac{T_{(i+1)}}{\Delta} \right\rfloor \Delta$, this implies that $T'_{(i+1)}$, $T'_{(i)}$, and $T'_{(i-1)}$ also do not collide with each other. *Therefore, $g(\mu)$ ensures that either User (i) will get selected or, if this is not the case, then one of the better users $(1), \ldots, (i-1)$ itself will get selected.* Therefore, the BS will be able to transmit data to the user selected by $g(\mu)$ at a rate that is greater than or equal to that of $f(\mu)$. Since this holds for each realization of timers, the average rate $g(\mu)$ will be greater than or equal to that of $f(\mu)$. Since $f(\mu)$ is optimal by assumption, the discrete mapping must also be optimal.

The proof for the other two cases, which correspond to: (i) $T_{(i+1)}$ expires before $T_{\max}$ and $i = 1$, and (ii) $T_{(i)}$ is the last timer to expire, is along similar lines.

B. Proof of Theorem 2

We prove the theorem in two steps.

Proposition 1: There exists an integer $1 \leq m \leq P$ such that $\sum_{i=0}^{N} \alpha^*[i] \leq 1 - \Gamma_m^{\text{eq}}$ and $\sum_{i=0}^{N} \alpha^*[i] > 1 - \Gamma_{m-1}^{\text{eq}}$.

Proof: Assume that such an m does not exist. Then, it follows that $1 - \Gamma_P^{\text{eq}} < \sum_{i=0}^{N} \alpha^*[i] \leq 1$. This implies that some timers from users whose metrics lie in the interval $[0, 1 - \Gamma_P^{\text{eq}})$ can expire. From Lemma 1, this is suboptimal. ■

The above result implies that there exist integers $r_1, \ldots, r_m$ such that exactly r_1 intervals lie entirely in the interval $(1 - \Gamma_1^{\text{eq}}, 1]$, and so on up to exactly $r_1 + \ldots + r_m = N + 1$ intervals lie within $(1 - \Gamma_m^{\text{eq}}, 1]$. Equivalently, $r_1, \ldots, r_m$ are the *largest* integers such that $\sum_{i=1}^{m} r_i = N + 1$ and $\sum_{i=0}^{i=r_1-1} \alpha^*[i] \triangleq s_1 \leq 1 - \Gamma_1^{\text{eq}}$, $\sum_{i=0}^{i=r_1+r_2-1} \alpha^*[i] \triangleq s_2 \leq 1 - \Gamma_2^{\text{eq}}$, ..., $\sum_{i=0}^{N} \alpha^*[i] \triangleq s_m \leq 1 - \Gamma_m^{\text{eq}}$.

Proposition 2: $\alpha^*[0] = \cdots = \alpha^*[r_1 - 1]$; $\alpha^*[r_1] = \cdots = \alpha^*[r_1 + r_2 - 1]$; ...; $\alpha^*\left[\sum_{i=1}^{m-1} r_i\right] = \cdots = \alpha^*[N]$.

Proof: The average rate expression in (4) is symmetric in $\alpha^*[0], \ldots, \alpha^*[r_1 - 1]$. This is because if a realization $\mathbf{x}$ results in a rate of R_1, then any permutation of the elements of the first r_1 elements of $\mathbf{x}$ also gives the same output. Therefore, the optimal values $\alpha^*[0], \ldots, \alpha^*[r_1 - 1]$ are equal. Similarly, we can show that (4) is symmetric in $\alpha^*[r_1], \ldots, \alpha^*[r_1 + r_2 - 1]$, and so on. Hence, the result. ■

The desired result in (5) then follows from Prop. 1, the definition of r_i, and Prop. 2.

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